What "are" row operations?

Ex:

$$\underbrace{\begin{bmatrix} 1 & 2 & 5 \\ -1 & 0 & 2 \\ 2 & 3 & -7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}}_{B}$$

$(3 \times \underbrace{3)(3}_{\text{match}} \times 2)$

result is (3×2)

What is the first row of AB?

(1,1)-entry:

$$\underbrace{a\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + c\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + e\begin{bmatrix} 5 \\ 2 \\ -7 \end{bmatrix}}_{\text{gives column 1 of } AB.}$$

notice the column of B gives "weights" for the columns of A

(1,1)-entry of AB is $1a + 2c + 5e$

(1,2)-entry is ... $1b + 2d + 5f$

so row 1 of AB is

$$= \begin{bmatrix} (1a + 2c + 5e) & (1b + 2d + 5f) \end{bmatrix}_{(1 \times 2)}$$

we see that this (2×1) is the same as:

$$\begin{aligned} = \; & 1\begin{bmatrix} a & b \end{bmatrix}_{(1 \times 2)} \\ + \; & 2\begin{bmatrix} c & d \end{bmatrix}_{(1 \times 2)} \\ + \; & 5\begin{bmatrix} e & f \end{bmatrix}_{(1 \times 2)} \end{aligned}$$

Similarly, row 2 of AB will be:

$$\begin{bmatrix} 1 & 2 & 5 \\ -1 & 0 & 2 \\ 2 & 3 & -7 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \longrightarrow \begin{aligned} & -1\begin{bmatrix} a & b \end{bmatrix} \\ + \; & 0\begin{bmatrix} c & d \end{bmatrix} \\ + \; & 2\begin{bmatrix} e & f \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} (-a + 2e) & (-b + 2f) \end{bmatrix}$$

row 3 is similarly made.

Thus, the i-th row of A gives "weights" for the rows of B in order to make the i-th row of AB.

Elementary row operations (by examples)

1) Swapping Rows : (Swap $R_2 \longleftrightarrow R_3$)

- (new R_1) = 1(old R_1)
- (new R_2) is = 1(old R_3)
- (new R_3) = 1(old R_2)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 3 & -3 & 0 \\ 2 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 7 & 9 \\ 3 & -3 & 0 \end{bmatrix}$$

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2) Scaling a row by c : (ex: $R_2 \to \frac{1}{3} R_2$)

(new R_2) = $\frac{1}{3}$(old R_2) $\longrightarrow$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 3 & -3 & 0 \\ 2 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 2 & 7 & 9 \end{bmatrix}$$

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3) Adding multiple of a row (Ex: $R_3 \to R_3 - 2R_1$)

(new R_3) = -2(old R_1) + 1(old R_3) $\longrightarrow$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 3 & -3 & 0 \\ 2 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -3 & 0 \\ 0 & 3 & 7 \end{bmatrix}$$

Thus, changing into echelon form or reduced echelon form is accomplished by chaining together multiplications of the types above:

<u>For example</u>: Reduce $\underline{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -3 & 0 \\ 2 & 7 & 9 \end{bmatrix}$ to (RREF)

$\begin{pmatrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{pmatrix}$ is the same as multiplying

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & -3 & 0 \\ 2 & 7 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & -9 & -3 \\ 0 & 3 & 7 \end{bmatrix}$$

$\left(R_2 \to -\frac{1}{9} R_2\right)$: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/9 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -9 & -3 \\ 0 & 3 & 7 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1/3 \\ 0 & 3 & 7 \end{bmatrix}$$

$(R_3 \to R_3 - 3R_2)$: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1/3 \\ 0 & 3 & 7 \end{bmatrix}$

echelon form $\longrightarrow$ $= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1/3 \\ 0 & 0 & 6 \end{bmatrix}$

So, in total, we see that

$$\left(\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&-3&1\end{bmatrix}}_{\underline{E_3}}\underbrace{\begin{bmatrix}1&0&0\\0&-1/9&0\\0&0&1\end{bmatrix}}_{\underline{E_2}}\underbrace{\begin{bmatrix}1&0&0\\-3&1&0\\-2&0&1\end{bmatrix}}_{\underline{E_1}}\right)\underbrace{\begin{bmatrix}1&2&1\\3&-3&0\\2&7&9\end{bmatrix}}_{\underline{A}}$$

$$= \begin{bmatrix}1&0&0\\-3&-1/9&0\\-2&-3&1\end{bmatrix}\underline{A} = \begin{bmatrix}1&2&1\\0&1&1/3\\0&0&6\end{bmatrix},$$

So multiplying $\underline{A}$ by this single matrix changes $\underline{A}$ to $(\underline{E}F)$. Similarly, we could go further and find a single matrix to change $\underline{A}$ into (RREF)

Summary below

Our two ways to multiply matrices (Summary)
(plus another simple, algorithmic approach)

1) Computing $\underline{AB}$ using the $\underline{\text{columns}}$ of $\underline{B}$ as "weights" for the $\underline{\text{columns}}$ of $\underline{A}$:

$$\underbrace{\begin{bmatrix} 1 & 2 & 5 \\ -1 & 0 & 2 \\ 2 & 3 & -7 \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 3 & -3 & 0 \\ 2 & 7 & 9 \end{bmatrix}}_{\underline{B}}$$

column 1 of $\underline{AB}$: $1\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 2 \\ -7 \end{bmatrix} = \begin{bmatrix} 17 \\ 3 \\ -3 \end{bmatrix}$

Column 2 of $\underline{AB}$: $2\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + 7\begin{bmatrix} 5 \\ 2 \\ -7 \end{bmatrix} = \begin{bmatrix} 31 \\ 12 \\ 54 \end{bmatrix}$

column 3 of $\underline{AB}$: $1\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + 9\begin{bmatrix} 5 \\ 2 \\ -7 \end{bmatrix} = \begin{bmatrix} 46 \\ 17 \\ -61 \end{bmatrix}$

so $\underline{AB} = \begin{bmatrix} 17 & 31 & 46 \\ 3 & 12 & 17 \\ -3 & -54 & -61 \end{bmatrix}$

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2) Computing $\underline{AB}$ using $\underline{\text{rows}}$ of $\underline{A}$ as "weights" for $\underline{\text{rows}}$ of $\underline{B}$:

$$\underbrace{\begin{bmatrix} 1 & 2 & 5 \\ -1 & 0 & 2 \\ 2 & 3 & -7 \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 3 & -3 & 0 \\ 2 & 7 & 9 \end{bmatrix}}_{\underline{B}}$$

row 1 of $\underline{AB}$:

$$\begin{aligned} &1\begin{bmatrix}1 & 2 & 1\end{bmatrix} \\ +\,&2\begin{bmatrix}3 & -3 & 0\end{bmatrix} \\ +\,&5\begin{bmatrix}2 & 7 & 9\end{bmatrix} \end{aligned} = \begin{bmatrix}17 & 31 & 46\end{bmatrix}$$

row 2 of $\underline{AB}$:

$$\begin{aligned} -&1\begin{bmatrix}1 & 2 & 1\end{bmatrix} \\ +\,&0\begin{bmatrix}3 & -3 & 0\end{bmatrix} \\ +\,&2\begin{bmatrix}2 & 7 & 9\end{bmatrix} \end{aligned} = \begin{bmatrix}3 & 12 & 17\end{bmatrix}$$

row 3 of $\underline{AB}$: $\cdots = \begin{bmatrix}25 & 44 & 65\end{bmatrix}$

so, again,

$$\underline{AB} = \begin{bmatrix} 17 & 31 & 46 \\ 3 & 12 & 17 \\ -3 & -54 & -61 \end{bmatrix}$$

A "third" method (but not really different)

$$\underbrace{\begin{bmatrix} 1 & 2 & 5 \\ -1 & 0 & 2 \\ 2 & 3 & -7 \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 3 & -3 & 0 \\ 2 & 7 & 9 \end{bmatrix}}_{\underline{B}} = \begin{bmatrix} & 31 & \\ & & 17 \\ & & \end{bmatrix}$$

"entrywise": the (i,j)-entry of $\underline{AB}$ is computed by pairing elements of <u>row i of $\underline{A}$</u> and <u>column j of $\underline{B}$</u>

<u>Ex</u>: $(1,2)$ entry $= 1(2) + 2(-3) + 5(7) = 31$

$(2,3)$ entry $= -1(1) + 0(0) + 2(9) = 17$

and so on ...